NEW DERIVED AUTOEQUIVALENCES OF HILBERT SCHEMES AND GENERALIZED KUMMER VARIETIES

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Abstract. We show that for every smooth projective surface X and every $n \geq 2$ the push-forward along the diagonal embedding gives a $\mathbb{P}^{n-1}$-functor into the $\mathfrak{S}_n$-equivariant derived category of X^n. Using the Bridgeland–King–Reid–Haiman equivalence this yields a new autoequivalence of the derived category of the Hilbert scheme of n points on X. In the case that the canonical bundle of X is trivial and $n = 2$ this autoequivalence coincides with the known EZ-spherical twist induced by the boundary of the Hilbert scheme. We also generalise the 16 spherical objects on the Kummer surface given by the exceptional curves to n^4 orthogonal $\mathbb{P}^{n-1}$-Objects on the generalised Kummer variety.

1. Introduction

For every smooth projective surface X over $\mathbb{C}$ and every $n \in \mathbb{N}$ there is the Bridgeland–King–Reid–Haiman equivalence (see [BKR01] and [Hai01])

$$\Phi\colon \mathrm{D}^b(X^{[n]}) \xrightarrow{\simeq} \mathrm{D}^b_{\mathfrak{S}_n}(X^n)$$

between the bounded derived category of the Hilbert scheme of n points on X and the $\mathfrak{S}_n$-equivariant derived category of the cartesian product of X. In [Plo07] Ploog used this to give a general construction which associates to every autoequvalence $\Psi \in \mathrm{Aut}(\mathrm{D}^b(X))$ an autoequivalence $\alpha(\Psi) \in \mathrm{Aut}(\mathrm{D}^b(X^{[n]}))$ on the Hilbert scheme. Recently, Ploog and Sosna [PS12] gave a construction that produces out of spherical objects (see [ST01]) on the surface $\mathbb{P}^n$-objects (see [HT06]) on $X^{[n]}$ which in turn induce further derived autoequivalences. On the other hand, there are only very few autoequivalences of $\mathrm{D}^b(X^{[n]})$ known to exist independently of $\mathrm{D}^b(X)$:

- There is always an involution given by tensoring with the alternating representation in $\mathrm{D}^b_{\mathfrak{S}_n}(X^n)$, i.e with the one-dimensional representation on which $\sigma \in \mathfrak{S}_n$ acts via multiplication by $\mathrm{sgn}(\sigma)$.
- Addington introduced in [Add11] the notion of a $\mathbb{P}^n$-functor generalising the $\mathbb{P}^n$-objects of Huybrechts and Thomas. He showed that for X a K3-surface and $n \geq 2$ the Fourier–Mukai transform $F_a\colon \mathrm{D}^b(X) \to \mathrm{D}^b(X^{[n]})$ induced by the universal sheaf is a $\mathbb{P}^{n-1}$-functor. This yields an autoequivalence of $\mathrm{D}^b(X^{[n]})$ for every K3-surface X and every $n \geq 2$.
- For $X = A$ an abelian surface the pull-back along the summation map $\Sigma\colon A^{[n]} \to A$ is a $\mathbb{P}^{n-1}$-functor and thus induces a derived autoequivalence (see [Mea12]).
- The *boundary of the Hilbert scheme* $\partial X^{[n]}$ is the codimension 1 subvariety of points representing non-reduced subschemes of X. For $n = 2$ it equals $X^{[2]}_\Delta := \mu^{-1}(\Delta)$ where $\mu\colon X^{[2]} \to S^2X$ denotes the Hilbert-Chow morphism. For $n = 2$ and X a surface with trivial canonical bundle it is known (see [Huy06, examples 8.49 (iv)]) that every line bundle on the boundary of the Hilbert scheme is an EZ-spherical object (see [Hor05])

and thus also induces an autoequivalence. We will see in remark 4.6 that the induced automorphisms given by different choices of line bundles on $X^{[2]}_\Delta$ only differ by twists with line bundles on $X^{[2]}$. Thus, we will just speak of *the* autoequivalence induced by the boundary referring to the automorphism induced by the EZ-spherical object $\mathcal{O}_{\mu|X^{[2]}_\Delta}(-1)$.

In this article we generalise this last example to surfaces with arbitrary canonical bundle and to arbitrary $n\geq 2$. More precisely, we consider the functor $F\colon \mathrm{D}^b(X)\to \mathrm{D}^b_{\mathfrak{S}_n}(X^n)$ which is defined as the composition of the functor $\mathrm{triv}\colon \mathrm{D}^b(X)\to \mathrm{D}^b_{\mathfrak{S}_n}(X)$ given by equipping every object with the trivial $\mathfrak{S}_n$-linearisation and the push-forward $\delta_*\colon \mathrm{D}^b_{\mathfrak{S}_n}(X)\to \mathrm{D}^b_{\mathfrak{S}_n}(X^n)$ along the diagonal embedding. Then we show in section 3 the following.

Theorem 1.1. *For every $n\in\mathbb{N}$ with $n\geq 2$ and every smooth projective surface X the functor $F\colon \mathrm{D}^b(X)\to \mathrm{D}^b_{\mathfrak{S}_n}(X^n)$ is a $\mathbb{P}^{n-1}$-functor.*

In section 4 we show that for $n=2$ the induced autoequivalence coincides under Φ with the autoequivalence induced by the boundary. In section 5 we compare the autoequivalence induced by F to some other derived autoequivalences of $X^{[n]}$ showing that it differs essentially from the standard autoequivalences and the autoequivalence induced by F_a. In particular, the Hilbert scheme always has non-standard autoequivalences even if X is a Fano surface. In the last section we consider the case that $X=A$ is an abelian surface. We show that after restricting our $\mathbb{P}^{n-1}$-functor from $A^{[n]}$ to the generalised Kummer variety $K_{n-1}A$ it splits into n^4 pairwise orthogonal $\mathbb{P}^{n-1}$-objects. They generalise the 16 spherical objects on the Kummer surface given by the line bundles $\mathcal{O}_C(-1)$ on the exceptional curves.

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2. $\mathbb{P}^n$-functors

A $\mathbb{P}^n$-*functor* is defined in [Add11] as a functor $F\colon\mathcal{A}\to\mathcal{B}$ of triangulated categories admitting left and right adjoints L and R such that

(i) There is an autoequivalence H of $\mathcal{A}$ such that
$$RF\simeq \mathrm{id}\oplus H\oplus H^2\oplus\cdots\oplus H^n\,.$$

(ii) The map
$$HRF\hookrightarrow RFRF\xrightarrow{R\varepsilon F} RF$$
with ε being the counit of the adjunction is, when written in the components
$$H\oplus H^2\oplus\cdots\oplus H^n\oplus H^{n+1}\to \mathrm{id}\oplus H\oplus H^2\oplus\cdots\oplus H^n\,,$$
of the form
$$\begin{pmatrix} * & * & \cdots & * & * \\ 1 & * & \cdots & * & * \\ 0 & 1 & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & * \end{pmatrix}.$$

(iii) $R \simeq H^n L$. If $\mathcal{A}$ and $\mathcal{B}$ have Serre functors, this is equivalent to $S_{\mathcal{B}} F H^n \simeq F S_{\mathcal{A}}$.

In the following we always consider the case that $\mathcal{A}$ and $\mathcal{B}$ are (equivariant) derived categories of smooth projective varieties and F is a Fourier–Mukai transform. The $\mathbb{P}^n$-*twist* associated to a $\mathbb{P}^n$-functor F is defined as the double cone

$$P_F := \operatorname{cone}\left(\operatorname{cone}(FHR \to FR) \to \mathrm{id}\right) .$$

The map defining the inner cone is given by the composition

$$FHR \xrightarrow{FjR} FRFR \xrightarrow{\varepsilon FR - FR\varepsilon} FR$$

where j is the inclusion given by the decomposition in (i). The map defining the outer cone is induced by the counit $\varepsilon\colon FR \to \mathrm{id}$ (for details see [Add11]). Taking the cones of the Fourier–Mukai transforms indeed makes sense, since all the occurring maps are induced by maps between the integral kernels (see [AL12]). We set $\ker R := \{B \in \mathcal{B} \mid RB = 0\}$. By the adjoint property it equals the right-orthogonal complement $(\operatorname{im} F)^{\perp}$.

Proposition 2.1 ([Add11, section 3])**.** *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a $\mathbb{P}^n$-functor.*

(i) *We have $P_F(B) = B$ for $B \in \ker R$.*
(ii) *$P_F \circ F \simeq H^{n+1}[2]$.*
(iii) *The objects in $\operatorname{im} F \cup \ker R$ form a spanning class of $\mathcal{B}$.*
(iv) *P_F is an autoequivalence.*

Example 2.2. (i) Let $\mathcal{B} = \mathrm{D}^b(X)$ for a smooth projective variety X. A $\mathbb{P}^n$-*object* (see [HT06]) is an object $E \in \mathcal{B}$ such that $E \otimes \omega_X \simeq E$ and $\operatorname{Ext}^*(E,E) \cong \mathrm{H}^*(\mathbb{P}^n, \mathbb{C})$ as $\mathbb{C}$-algebras (the ring structure on the left-hand side is the Yoneda product and on the right-hand side the cup product). A $\mathbb{P}^n$-object can be identified with the $\mathbb{P}^n$-functor

$$F\colon \mathrm{D}^b(\mathrm{pt}) \to \mathcal{B} \quad , \quad \mathbb{C} \mapsto E$$

with $H = [-2]$. Note that the right adjoint is indeed given by $R = \operatorname{Ext}^*(E, \_)$. The $\mathbb{P}^n$-twist associated to the functor F is the same as the $\mathbb{P}^n$-twist associated to the object E as defined in [HT06].

(ii) A $\mathbb{P}^1$-functor is the same as a *spherical functor* (see [Ann07]) where the unit

$$\mathrm{id} \xrightarrow{\eta} RF \to H$$

splits. In this case there is also the *spherical twist* given by

$$T_F := \operatorname{cone}\left(FR \xrightarrow{\varepsilon} \mathrm{id}\right) .$$

It is again an autoequivalence with $T_F^2 = P_F$ (see [Add11, p. 33]).

Lemma 2.3. (i) *Let $\Psi \in \operatorname{Aut}(\mathcal{A})$ such that $\Psi \circ H \simeq H \circ \Psi$. Then $F \circ \Psi$ is again a $\mathbb{P}^n$-functor with the property*

$$P_{F\circ\Psi} \simeq P_F .$$

(ii) *Let $\Phi\colon \mathcal{B} \to \mathcal{C}$ be an equivalence of triangulated categories. Then $\Phi \circ F$ is again a $\mathbb{P}^n$-functor with the property that*

$$P_{\Phi\circ F} \circ \Phi \simeq \Phi \circ P_F .$$

Proof. The proof is analogous to the proof of the corresponding statement for spherical functors [Ann07, proposition 2]. □

Corollary 2.4. *Let $E_1,\dots,E_n \in \mathcal{B}$ be a collection of pairwise orthogonal (that means $\mathrm{Hom}^*(E_i,E_j)=0=\mathrm{Hom}^*(E_j,E_i)$ for $i\neq j$) $\mathbb{P}^n$-objects with associated twists $p_i := P_{E_i}$. Then*

$$\mathbb{Z}^n \to \mathrm{Aut}(\mathcal{A}) \quad , \quad (\lambda_1,\dots,\lambda_n)\mapsto p_1^{\lambda_1}\circ\dots\circ p_n^{\lambda_n}$$

defines a group isomorphism $\mathbb{Z}^n \cong \langle p_1,\dots,p_n\rangle \subset \mathrm{Aut}(\mathcal{B})$.

Proof. By part (ii) of the previous lemma the p_i commute which means that the map is indeed a group homomorphism onto the subgroup generated by the p_i. Let $g = p_1^{\lambda_1}\circ\dots\circ p_n^{\lambda_n}$. Then $g(E_i) = E_i[2n\lambda_i]$ by proposition 2.1. Thus, $g=\mathrm{id}$ implies $\lambda_1=\dots=\lambda_n=0$. □

Lemma 2.5. *Let X be a smooth variety, $T\in \mathrm{Aut}(\mathrm{D}^b(X))$, and $A,B\in \mathrm{D}^b(X)$ objects such that $TA=A[i]$ and $TB=B[j]$ for some $i\neq j\in\mathbb{Z}$. Then $A\perp B$ and $B\perp A$.*

Proof. See [Add11, p. 11]. □

Remark 2.6. This shows together with proposition 2.1 that for a $\mathbb{P}^n$-functor F with $H=[-\ell]$ for some $\ell\in\mathbb{Z}$ there does not exist a non zero-object A with $T_F(A)=A[i]$ for any values of i besides 0 and $-n\ell+2$ because such an object would be orthogonal to the spanning class $\mathrm{im}\, F\cup \ker R$.

3. The diagonal embedding

Let X be a smooth projective surface over $\mathbb{C}$ and $2\le n\in\mathbb{N}$. We denote by $\delta\colon X\to X^n$ the diagonal embedding. We want to show that $F\colon \mathrm{D}^b(X)\to \mathrm{D}^b_{\mathfrak{S}_n}(X^n)$ given as the composition

$$\mathrm{D}^b(X)\xrightarrow{\mathrm{triv}}\mathrm{D}^b_{\mathfrak{S}_n}(X)\xrightarrow{\delta_*}\mathrm{D}^b_{\mathfrak{S}_n}(X^n)$$

of the functor which maps each object to itself equipped with the trivial action and the equivariant push-forward is a $\mathbb{P}^{n-1}$-functor. Its right adjoint R is given as the composition

$$\mathrm{D}^b_{\mathfrak{S}_n}(X^n)\xrightarrow{\delta^!}\mathrm{D}^b_{\mathfrak{S}_n}(X)\xrightarrow{[\_]^{\mathfrak{S}_n}}\mathrm{D}^b(X^n)$$

of the usual right adjoint (see [LH09] for equivariant Grothendieck duality) and the functor of taking invariants. We consider the standard representation ϱ of $\mathfrak{S}_n$ as the quotient of the regular representation $\mathbb{C}^n$ by the one dimensional invariant subspace. The normal bundle sequence

$$0\to T_X\to T_{X^n|X}\to N\to 0$$

where $N:=N_\delta = N_{X/X^n}$ is of the form

$$0\to T_X\to T_X^{\oplus n}\to N\to 0$$

where the map $T_X\to T_X^{\oplus n}$ is the diagonal embedding. When considering $T_{X^n|X}$ as a $\mathfrak{S}_n$-sheaf equipped with the natural linearisation it is given by $T_X\otimes\mathbb{C}^n$ where $\mathbb{C}^n$ is the regular representation. Thus, as a $\mathfrak{S}_n$-sheaf, the normal bundle equals $T_X\otimes\varrho$. We also see that the normal bundle sequence splits using e.g. the splitting

$$T_X\otimes\mathbb{C}^n\to T_X \quad , \quad (v_1,\dots,v_n)\mapsto \frac{1}{n}(v_1+\dots+v_n)\,.$$

Theorem 3.1 ([AC12])**.** *Let $\iota\colon Z\hookrightarrow M$ be a regular embedding of codimension c such that the normal bundle sequence splits. Then there is an isomorphism*

$$\iota^*\iota_*(\_)\simeq(\_)\otimes\Big(\bigoplus_{i=0}^{c}\wedge^i N^\vee_{Z/M}[i]\Big) \tag{1}$$

of endofunctors of $\mathrm{D}^b(Z)$.

Corollary 3.2. *Under the same assumptions, there is an isomorphism*

$$\iota^!\iota_*(\_) \simeq (\_) \otimes (\bigoplus_{i=0}^{c} \wedge^i N_{Z/M}[-i]) \tag{2}$$

Proof. Tensorise both sides of (1) by $\iota^!\mathcal{O}_M \simeq \wedge^c N_{Z/M}[-c]$. □

Lemma 3.3. *The monad multiplication* $\iota^!\varepsilon\iota_* \colon \iota^!\iota_*\iota^!\iota_* \to \iota^!\iota_*$ *is given under the above isomorphism (if chosen correctly) by the wedge pairing*

$$(\bigoplus_{i=0}^{c} \wedge^i N_{Z/M}[-i]) \otimes (\bigoplus_{j=0}^{c} \wedge^j N_{Z/M}[-j]) \to \bigoplus_{k=0}^{c} \wedge^k N_{Z/M}[-k]\,.$$

Proof. For $E \in \mathrm{D}^b(M)$ the object $\iota^! E$ can be identified with $\mathcal{H}om(\iota_*\mathcal{O}_Z, E)$ considered as an object in $\mathrm{D}^b(Z)$. Under this identification the counit map $\mathcal{H}om(\iota_*\mathcal{O}_Z, E) \to E$ is given by $\varphi \mapsto \varphi(1)$ (see [Har66, section III.6]). Now we get for $F \in \mathrm{D}^b(Z)$ the identifications

$$\iota^!\iota_*\iota^!\iota_* F \simeq \mathcal{H}om(\iota_*\mathcal{O}_Z, \iota_*\mathcal{O}_Z) \otimes_{\mathcal{O}_Z} \mathcal{H}om(\iota_*\mathcal{O}_Z, \iota_*\mathcal{O}_Z) \otimes_{\mathcal{O}_Z} F$$

and $\iota^!\iota_* F \simeq \mathcal{H}om(\iota_*\mathcal{O}_Z, \iota_*\mathcal{O}_Z) \otimes_{\mathcal{O}_Z} F$ under which the monad multiplication equals the Yoneda product. It is known (see [LH09, p. 442]) that the Yoneda product corresponds to the wedge product when choosing the right isomorphism. □

In the case that $\iota = \delta$ from above this yields the isomorphism of monads

$$\delta^!\delta_*(\_) \simeq (\_) \otimes (\bigoplus_{i=0}^{2(n-1)} \wedge^i (T_X \otimes \varrho)[-i])\,. \tag{3}$$

Lemma 3.4 ([Sca09a, Appendix B])**.** *Let* V *be a two-dimensional vector space with a basis consisting of vectors* u *and* v*. Then the space of invariants* $[\wedge^i(V \otimes \varrho)]^{\mathfrak{S}_n}$ *is one-dimensional if* $0 \le i \le 2(n-1)$ *is even and zero if it is odd. In the even case* $i = 2\ell$ *the space of invariants is spanned by the image of the vector* ω^ℓ*, where*

$$\omega = \sum_{i=1}^{k} ue_i \wedge ve_i \in \wedge^2(V \otimes \mathbb{C}^n)\,,$$

under the projection induced by the projection $\mathbb{C}^n \to \varrho$.

Corollary 3.5. *For a vector bundle* E *on* X *of rank two and* $0 \le \ell \le n-1$ *there is an isomorphism*

$$[\wedge^{2\ell}(E \otimes \varrho)]^{\mathfrak{S}_n} \cong (\wedge^2 E)^{\otimes \ell}\,.$$

Proof. The isomorphism is given by composing the morphism

$$(\wedge^2 E)^{\otimes \ell} \to \wedge^\ell(E \otimes \mathbb{C}^n) \quad , \quad x_1 \otimes \cdots \otimes x_\ell \mapsto \sum_{1 \le i_1 < \cdots < i_\ell \le n} x_1 e_{i_1} \wedge \cdots \wedge x_\ell e_{i_\ell}$$

with the projection induced by the projection $\mathbb{C}^n \to \varrho$. □

We set $H := \wedge^2 T_X[-2] = \omega_X^\vee[-2] = S_X^{-1}$.

Corollary 3.6. *There is the isomorphism of functors*

$$RF \simeq \mathrm{id} \oplus H \oplus H^2 \oplus \cdots \oplus H^{n-1}\,.$$

Proof. This follows by formula (3) and corollary 3.5. ☐

Lemma 3.7. *The map $HRF \to RF$ defined in condition (ii) for $\mathbb{P}^n$-functors is for this pair $F \rightleftharpoons R$ given by the matrix*

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Proof. The generators ω^ℓ from lemma 3.4 are mapped to each other by wedge product. By lemma 3.3 the monad multipliction is given by wedge product. ☐

Lemma 3.8. *There is the isomorphism $S_{X^n}FH^{n-1} \simeq FS_X$.*

Proof. For $\mathcal{E} \in \mathrm{D}^b(X)$ there are natural isomorphisms

$$\begin{aligned} S_{X^n}FH^{n-1}(\mathcal{E}) = \omega_{X^n}[2n] \otimes \delta_*(\mathcal{E} \otimes \omega_X^{-(n-1)}[-2(n-1)]) &\simeq \omega_X^{\boxtimes n} \otimes \delta_*(\mathcal{E} \otimes \omega_X^{-(n-1)})[2] \\ &\overset{\mathrm{PF}}{\simeq} \delta_*(\mathcal{E} \otimes \omega_X[2]) = FS_X(\mathcal{E})\,. \end{aligned}$$

☐

All this together shows theorem 1.1, i.e. that $F = \delta_* \circ \mathrm{triv}$ is indeed a $\mathbb{P}^{n-1}$-functor.

4. Composition with the Bridgeland–King–Reid–Haiman equivalence

The *isospectral Hilbert scheme* $I^nX \subset X^{[n]} \times X^n$ is defined as the reduced fibre product $I^nX := \left(X^{[n]} \times_{S^nX} X^n\right)_{\mathrm{red}}$ with the defining morphisms being the Hilbert–Chow morphism $\mu\colon X^{[n]} \to S^nX$ and the quotient morphism $\pi\colon X^n \to S^nX$. Thus, there is the commutative diagram

$$\begin{array}{ccc} I^nX & \xrightarrow{q} & X^n \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ X^{[n]} & \xrightarrow[\mu]{} & S^nX\,. \end{array}$$

The *Bridgeland–King–Reid–Haiman equivalence* is the functor

$$\Phi := \mathrm{FM}_{\mathcal{O}_{I^n_X}} \circ \mathrm{triv} = p_* \circ q^* \circ \mathrm{triv}\colon\ \mathrm{D}^b(X^{[n]}) \longrightarrow \mathrm{D}^b_{\mathfrak{S}_n}(X^n)\,.$$

By the results in [BKR01] and [Hai01] it is indeed an equivalence. The isospectral Hilbert scheme can be identified with the blow-up of X^n along the union of all the pairwise diagonals $\Delta_{ij} = \{(x_1, \dots, x_n) \in X^n \mid x_i = x_j\}$ (see [Hai01]). By lemma 2.3 the functor composition $\Phi^{-1} \circ F\colon \mathrm{D}^b(X) \to \mathrm{D}^b(X^{[n]})$ is again a $\mathbb{P}^n$-functor and thus yields an autoequivalence of the derived category of the Hilbert scheme.

Lemma 4.1. *Let $G\colon \mathcal{A} \to \mathcal{B}$ be a right-exact functor between abelian categories such that $\mathcal{A}$ has a G-adapted class and let $X^\bullet \in \mathrm{D}^-(\mathcal{A})$ be a complex such that $\mathcal{H}^q(X^\bullet)$ is G-acyclic for every $n \in \mathbb{Z}$. Then $\mathcal{H}^n(LG(X^\bullet)) = G\mathcal{H}^n(X^\bullet)$ holds for every $n \in \mathbb{Z}$.*

Proof. This follows from the spectral sequence

$$E_2^{p,q} = L^pG\mathcal{H}^q(X^\bullet) \Longrightarrow E^n = \mathcal{H}^n(LG(X^\bullet))\,.$$

☐

If the surface X has trivial canonical bundle, it is known that any line bundle L on the boundary $\partial X^{[2]} = X^{[2]}_\Delta$ of the Hilbert scheme of two points on X is an EZ-spherical object (see [Huy06, examples 8.49 (iv)]). That means that the functor

$$\tilde{F}_L\colon \mathrm{D}^b(X) \to \mathrm{D}^b(X^{[2]}) \quad , \quad \mathcal{E} \mapsto j_*(L \otimes \mu_\Delta^* E)$$

is a spherical functor where the maps j and μ_Δ come from the fibre diagram

$$\begin{tikzcd} X^{[2]}_\Delta \arrow[r, "j"] \arrow[d, "\mu_\Delta"'] & X^{[2]} \arrow[d, "\mu"] \\ X \arrow[r, "d"'] & S^2X \end{tikzcd}$$

with d being the diagonal embedding. The map μ_Δ is a $\mathbb{P}^1$-bundle.

Proposition 4.2. *Let X be a smooth projective surface (with arbitrary canonical bundle). Then there is an isomorphism of functors $\Phi^{-1} \circ F \simeq \tilde{F}_{\mathcal{O}_{\mu_\Delta}(-1)}$, where Φ^{-1} is the inverse of the BKRH-equivalence and $F = \delta_* \circ \mathrm{triv}\colon \mathrm{D}^b(X) \to \mathrm{D}^b_{\mathfrak{S}_2}(X^2)$ from the previous section.*

Proof. The functor Φ^{-1} is given by the composition $[\_]^{\mathfrak{S}_2} \circ \mathrm{FM}_{\mathcal{Q}}^{X^2 \to X^{[2]}}$ with Fourier–Mukai kernel $\mathcal{Q} = \mathcal{O}^\vee_{I^2X} \otimes q^*\omega_{X^{[2]}}[4]$. The isospectral Hilbert scheme I^2X is the blow-up of X^2 along the diagonal. In particular, it is smooth. Let $E = p^{-1}(\Delta)$ be the exceptional divisor of the blow up. The $\mathbb{P}^1$-bundles $p_\Delta\colon E \to X$ and $\mu_\Delta\colon X^{[2]}_\Delta \to X$ are isomorphic via $q\colon I^2X \to X^{[2]}$. The canonical bundle of the blow-up is given by $\omega_{I^2X} \cong p^*\omega_{X^2} \otimes \mathcal{O}(E)$. Let N be the normal bundle of the codimension 4 regular embedding $I^2X \to X^{[2]} \times X^2$. By adjunction formula

$$\wedge^4 N \cong \omega^\vee_{X^{[2]}\times X^2 | I^2X} \otimes \omega_{I^2X} \cong q^*\omega^\vee_{X^{[2]}} \otimes \mathcal{O}(E)\,.$$

It follows by Grothendieck-Verdier duality for regular embeddings that

$$\mathcal{Q} = \mathcal{O}^\vee_{I^2X} \otimes q^*\omega_{X^{[2]}}[4] \simeq \wedge^4 N[-4] \otimes q^*\omega_{X^{[2]}}[4] \simeq \mathcal{O}(E)\,.$$

Here, the line bundle $\mathcal{O}(E)$ is equipped with the natural $\mathfrak{S}_2$-linearisation which is trivial over E, i.e. $\mathcal{O}_E(E) = \mathcal{O}_{p_\Delta}(-1)$ carries the trivial $\mathfrak{S}_2$-action. We need the following slight generalisation of [Huy06, Proposition 11.12] for a blow-up

$$\begin{tikzcd} E \arrow[r, "i"] \arrow[d, "\pi"'] & \tilde{X} \arrow[d, "p"] \\ Y \arrow[r, "j"'] & X \end{tikzcd}$$

of a smooth projective variety X along a smooth subvariety Y of codimension c.

Lemma 4.3. *For every $\mathcal{F} \in \mathrm{Coh}(Y)$ and every $k \in \mathbb{Z}$ there is an isomorphism*

$$\mathcal{H}^k(p^*j_*\mathcal{F}) \cong i_*\left(\pi^*\mathcal{F} \otimes \wedge^{-k}\Omega_\pi \otimes \mathcal{O}_\pi(-k)\right)\,.$$

Proof. This can be proven locally. Hence, we may assume that $Y = Z(s)$ is the zero locus of a global section of a vector bundle $\mathcal{E}$ of rank c. Thus, the blow-up diagram can be enlarged

to

$$\begin{array}{ccccc} E & \xrightarrow{i} & \tilde{X} & \xrightarrow{\iota} & \mathbb{P}(\mathcal{E}) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[j]{} & X & \xrightarrow[\mathrm{id}]{} & X \end{array}$$

where ι is a closed embedding of codimension $c-1$ such that the normal bundle M has the property $\wedge^k M^\vee_{|E} = \wedge^{-k}\Omega_\pi \otimes \mathcal{O}_\pi(-k)$ (see [Huy06, p. 252]). The outer square is a flat base change. It follows that

$$\iota_* p^* j_* \mathcal{F} \simeq \iota_*\iota^* g^* j_*\mathcal{F} \simeq \iota_*\iota^*\iota_* i_*\pi^*\mathcal{F} \simeq \iota_* i_*(\pi^*\mathcal{F}\otimes i^*\iota^*\iota_*\mathcal{O}_{\tilde{X}}) \tag{4}$$

where the last isomorphism is given by applying the projection formula two times. Now $\mathcal{H}^k(\iota^*\iota_*\mathcal{O}_{\tilde{X}}) \cong \wedge^{-k}M^\vee$ is locally free for every $k \in \mathbb{Z}$. By lemma 4.1 it follows that

$$\mathcal{H}^k(\pi^*\mathcal{F}\otimes i^*\iota^*\iota_*\mathcal{O}_{\tilde{X}}) \cong \pi^*\mathcal{F}\otimes i^*\mathcal{H}^k(\iota^*\iota_*\mathcal{O}_{\tilde{X}}) \cong \pi^*\mathcal{F}\otimes \wedge^k M^\vee_{|E} \cong \pi^*\mathcal{F}\otimes\wedge^{-k}\Omega_\pi\otimes\mathcal{O}_\pi(-k)\,.$$

By (4) also

$$\iota_*\mathcal{H}^k(p^*j_*\mathcal{F}) \cong \mathcal{H}^k(\iota_*p^*j_*\mathcal{F}) \cong \mathcal{H}^k\left(\iota_* i_*(\pi^*\mathcal{F}\otimes i^*\iota^*\iota_*\mathcal{O}_{\tilde{X}})\right) \cong \iota_* i_*\mathcal{H}^k(\pi^*\mathcal{F}\otimes i^*\iota^*\iota_*\mathcal{O}_{\tilde{X}})$$

which proves the assertion since $\iota_*\colon \mathrm{Coh}(\tilde{X}) \to \mathrm{Coh}(\mathbb{P}(\mathcal{E}))$ is fully faithful. □

Remark 4.4. If X carries an action by a finite group G and Y is invariant under this action, G also acts on the blow-up $\tilde{X}$. The bundle $M_{|E}$ of the proof is a quotient of the normal bundle $N_{E/\mathbb{P}(\mathcal{E})} \cong \pi^*N_{Y/X}$. In the case that there is a group action this quotient is G-equivariant. Thus, the formula of the lemma is in this case also true for $\mathcal{F}\in \mathrm{Coh}_G(X)$ with the action on the right hand side induced by the linearization of the wedge powers of M respectively $N_{Y/X}$.

In the case of the blow-up $p\colon I^2X \to X^2$ the center Δ of the blow-up has codimension 2. Thus, $p^*\delta_*\mathcal{F}$ is cohomologically concentrated in degree 0 and -1. Since $p^*\delta_*\mathcal{F}$ is concentrated on E where $\mathfrak{S}_2$ is acting trivially, one can take the invariants even before applying the push-forward along $q\colon I^2X \to X^{[2]}$. The group $\mathfrak{S}_2$ acts on $\wedge^0 N_{\Delta/X^2}$ trivially and on N_{Δ/X^2} alternating. Hence, by the previous remark we have for $\mathcal{F}\in\mathrm{Coh}(X)$ equipped with the trivial group action that $\mathcal{H}^0(p^*\delta_*\mathcal{F})^{\mathfrak{S}_2} = p_\Delta^*\mathcal{F}$ and $\mathcal{H}^1(p^*\delta_*\mathcal{F})^{\mathfrak{S}_2} = 0$. In particular, every $\mathcal{F}\in\mathrm{Coh}(X)$ is acyclic under the functor $[\_]^{\mathfrak{S}_2}\circ p^*\circ\delta_*\circ\mathrm{triv}$ with $[\_]^{\mathfrak{S}_2}\circ p^*\circ\delta_*\circ\mathrm{triv}(\mathcal{F}) \simeq p_\Delta^*(\mathcal{F})$. This implies that $[\_]^{\mathfrak{S}_2}\circ p^*\circ\delta_*\circ\mathrm{triv}(\mathcal{F}^\bullet) \simeq p_\Delta^*(\mathcal{F}^\bullet)$ for every $\mathcal{F}^\bullet\in \mathrm{D}^b(X)$. Together with $\mathcal{Q}_{|E}\cong\mathcal{O}_E(E)\cong\mathcal{O}_{p_\Delta}(-1)\cong\mathcal{O}_{\mu_\Delta}(-1)$ this proves proposition 4.2. □

Remark 4.5. The proposition says in particular that $\tilde{F}_{\mathcal{O}_{\mu_\Delta}(-1)}$ is also a spherical functor in the case that ω_X is not trivial. One can also prove this directly and for general L instead of $\mathcal{O}_{\mu_\Delta}(-1)$.

Remark 4.6. Since $X^{[2]}_\Delta$ is a $\mathbb{P}^1$-bundle over X, every line bundle on it is of the form $L\cong\mu_\Delta^*K\otimes\mathcal{O}_{\mu_\Delta}(i)$ for some $K\in\mathrm{Pic}\,X$ and $i\in\mathbb{Z}$. The canonical bundle of $X^{[2]}_\Delta$ is given by $\mu_\Delta^*\omega_X\otimes\mathcal{O}_{\mu_\Delta}(-2)$. The Hilbert-Chow morphism μ is a crepant resolution, i.e. $\omega_{X^{[2]}}\cong\mu^*\omega_{S^2X}$. Thus,

$$\omega_{X^{[2]}|X^{[2]}_\Delta} \cong \mu_\Delta^*(\omega_{S^2X|\Delta}) \cong \mu_\Delta^*\omega_X^2\,.$$

Let $N = \mathcal{O}_{X^{[n]}_\Delta}(X^{[n]}_\Delta)$ be the normal bundle of $X^{[2]}_\Delta$ in $X^{[2]}$. By adjunction formula it is given by $\mu_\Delta^*\omega_X^\vee\otimes\mathcal{O}_{\mu_\Delta}(-2)$. There is a line bundle $D\in\mathrm{Pic}\,X^{[2]}$ (namely the determinant

of the tautological sheaf $\mathcal{O}_X^{[2]}$) such that $-2c_1(D) = [X_\Delta^{[2]}] = c_1(\mathcal{O}(X_\Delta^{[2]}))$ (see [Leh99, lemma 3.8]). Its restriction $D_{|X_\Delta^{[2]}}$ is of the form $\mu_\Delta^* M \otimes \mathcal{O}_{\mu_\Delta}(1)$ for some $M \in \operatorname{Pic} X$. Using this, we can rewrite for a general $L = \mu_\Delta^* K \otimes \mathcal{O}_{\mu_\Delta}(i) \in \operatorname{Pic} X_\Delta^{[2]}$ the spherical functor $\tilde{F}_L$ as $\tilde{F}_L = M_D^{i+1} \circ \tilde{F}_{\mathcal{O}_{\mu_\Delta(-1)}} \circ M_Q$ for some $Q \in \operatorname{Pic} X$ where M_Q is the autoequivalence given by tensor product with Q. The analogous of lemma 2.3 for spherical functors thus yields $t_L = M_D^{i+1} \circ t_{\mathcal{O}_{\mu_\Delta}(-1)} \circ M_D^{-(i+1)}$ where t_L is the spherical twist associated to $\tilde{F}_L$.

Remark 4.7. For general $n \geq 2$ every object in the image of $\Phi^{-1} \circ F$ still is supported on $X_\Delta^{[n]} = \mu^{-1}(\Delta)$.

5. Comparison with other autoequivalences

In the following we will denote the $\mathbb{P}^n$-twist associated to F respectively $\Phi^{-1} \circ F$ by $b \in \operatorname{Aut}(\mathrm{D}^b_{\mathfrak{S}_n}(X^n)) \cong \operatorname{Aut}(\mathrm{D}^b(X^{[n]}))$. In the case that $n = 2$ the functor F is spherical (see example 2.2 (ii)). We denote the associated spherical twist by $\sqrt{b}$.

Proposition 5.1. *The automorphism b is not contained in the group of standard automorphisms*

$$\operatorname{Aut}(\mathrm{D}^b(X^{[n]})) \supset \mathrm{DAut}_{st}(X^{[n]}) \cong \mathbb{Z} \times \Big(\operatorname{Aut}(X^{[n]}) \ltimes \operatorname{Pic}(X^{[n]})\Big)$$

generated by shifts, push-forwards along automorphisms and taking tensor products by line bundles. The same holds in the case $n = 2$ for $\sqrt{b}$.

Proof. Let $[\xi] \in X^{[n]} \setminus X_\Delta^{[n]}$, i.e. $|\operatorname{supp} \xi| \geq 2$. Then by remark 4.7 and proposition 2.1 (i), we have $b(\mathbb{C}([\xi])) = \mathbb{C}([\xi])$. Let $g = [\ell] \circ \varphi_* \circ M_L \in \mathrm{DAut}_{st}(X^{[n]})$ where M_L is the functor $E \mapsto E \otimes L$ for an $L \in \operatorname{Pic} X^{[n]}$. Then $g(\mathbb{C}([\xi])) = \mathbb{C}(\varphi([\xi]))[\ell]$. Thus, the assumption $b = g$ implies $\ell = 0$ and also $\varphi = \mathrm{id}$, since $X^{[n]} \setminus X_\Delta^{[n]}$ is open in $X^{[n]}$. Thus, the only possibility left for $b \in \mathrm{DAut}_{st}(X^{[n]})$ is $b = M_L$ for some line bundle L which can not hold by proposition 2.1 (ii). The proof that $\sqrt{b} \notin \mathrm{DAut}_{st}(X^{[n]})$ is the same. □

In [Plo07] Ploog gave a general construction which associates to derived autoequivalences of the surface X derived autoequivalences of the Hilbert scheme $X^{[n]}$. Let $\Psi \in \operatorname{Aut}(\mathrm{D}^b(X))$ with Fourier–Mukai kernel $\mathcal{P} \in \mathrm{D}^b(X \times X)$. The object $\mathcal{P}^{\boxtimes n} \in \mathrm{D}^b(X^n \times X^n)$ carries a natural $\mathfrak{S}_n$-linearisation given by permutation of the box factors. Thus, it induces a $\mathfrak{S}_n$-equivariant derived autoequivalence $\alpha(\Psi) := \mathrm{FM}_{\mathcal{P}^{\boxtimes n}}$ of X^n. This gives the following.

Theorem 5.2 ([Plo07])**.** *The above construction gives an injective group homomorphism*

$$\alpha\colon \operatorname{Aut}(\mathrm{D}^b(X)) \to \operatorname{Aut}(\mathrm{D}^b_{\mathfrak{S}_n}(X^n)) \cong \operatorname{Aut}(\mathrm{D}^b(X^{[n]}))\,.$$

Remark 5.3. For every $\varphi \in \operatorname{Aut}(X)$ we have $\alpha(\varphi_*) = (\varphi^n)_*$ where φ^n is the $\mathfrak{S}_n$-equivariant automorphism of X^n given by $\varphi(x_1, \dots, x_n) = (\varphi(x_1), \dots, \varphi(x_n))$. Furthermore, φ acts on $X^{[n]}$ by the morphism $\varphi^{[n]}$, which is given by $\varphi^{[n]}([\xi]) = [\varphi(\xi)]$, and on X^n by the morphism φ^n. Since the Bridgeland–King–Reid–Haiman equivalence is the Fourier–Mukai transform with kernel the structural sheaf of I^nX, it is $\operatorname{Aut}(X)$-equivariant, i.e. $\Phi \circ (\varphi^{[n]})_* \simeq (\varphi^n)_* \circ \Phi$. Thus, $\alpha(\varphi_*) \in \operatorname{Aut}(\mathrm{D}^b_{\mathfrak{S}_n}(X^n))$ corresponds to $\varphi_*^{[n]} \in \operatorname{Aut}(\mathrm{D}^b(X^{[n]}))$. For $L \in \operatorname{Pic} X$ we have $\alpha(M_L) = M_{L^{\boxtimes n}}$ where $L^{\boxtimes n}$ is considered as a $\mathfrak{S}_n$-equivariant line bundle with the natural linearization. Under Φ the automorphism $M_{L^{\boxtimes n}}$ corresponds to $M_{\mathcal{D}_L} \in \operatorname{Aut}(\mathrm{D}^b(X^{[n]}))$ where $\mathcal{D}_L \in \operatorname{Pic} X^{[n]}$ is the line bundle $\mathcal{D}_L := \mu^*((L^{\boxtimes n})^{\mathfrak{S}_n})$ (see [Kru12, lemma 9.2]).

Lemma 5.4. (i) *For every automorphism $\varphi \in \mathrm{Aut}(X)$ we have $b \circ \alpha(\varphi_*) = \alpha(\varphi_*) \circ b$ and for $n = 2$ also $\sqrt{b} \circ \alpha(\varphi_*) = \alpha(\varphi_*) \circ \sqrt{b}$.*

(ii) *For every line bundle $L \in \mathrm{Pic}(X)$ we have $b \circ \alpha(M_L) = \alpha(M_L) \circ b$ and for $n = 2$ also $\sqrt{b} \circ \alpha(M_L) = \alpha(M_L) \circ \sqrt{b}$.*

Proof. We have $\alpha(\varphi_*) \circ F \simeq F \circ \varphi_*$ and $\alpha(M_L) \circ F \simeq F \circ M_L^n$. The assertions now follow by lemma 2.3 (for $\sqrt{b}$ one has to use the analogous result [Ann07, proposition 2] for spherical twists). □

Let $G \subset \mathrm{Aut}(\mathrm{D}^b(X^{[n]}))$ be the subgroup generated by b, shifts, and $\alpha(\mathrm{DAut}_{st}(X))$.

Proposition 5.5. *The map*

$$S\colon \mathbb{Z} \times \mathbb{Z} \times (\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X)) \to \mathrm{Aut}(\mathrm{D}^b_{\mathfrak{S}_n}(X^n)) \quad , \quad (k, \ell, \Psi) \mapsto b^k \circ [\ell] \circ \alpha(\Psi)$$

defines a group isomorphism onto G.

Proof. By the previous lemma, b indeed commutes with $\alpha(\Psi)$ for $\Psi \in \mathrm{DAut}_{st}(X)$. Together with theorem 5.2 and the fact that shifts commute with every derived automorphism, this shows that S is indeed a well-defined group homomorphism with image G. Now consider $g = b^k \circ [\ell] \circ \alpha(\varphi_*) \circ \alpha(M_L)$ and assume $g = \mathrm{id}$. For every point $[\xi] \in X^{[n]} \setminus X^{[n]}_\Delta$ we have $g(\mathbb{C}([\xi])) = \mathbb{C}([\varphi(\xi)])[\ell]$ which shows $\ell = 0$ and $\varphi = \mathrm{id}$, i.e. $g = b^k \circ M_{L^{\boxtimes n}}$. Hence, for $A \in \mathrm{D}^b(X)$ its image under F gets mapped to $g(FA) = F(A \otimes N)[k(2n-2)]$ for some line bundle N on X, which shows that $k = 0$. Finally, $g = M_{L^{\boxtimes n}}$ is trivial only if $L = \mathcal{O}_X$. □

Remark 5.6. Again, the analogous statement with b replaced by $\sqrt{b}$ holds.

Let now X be a K3-surface. In this case Addington has shown in [Add11] that the Fourier–Mukai transform $F_a\colon \mathrm{D}^b(X) \to \mathrm{D}^b(X^{[n]})$ with kernel the universal sheaf $\mathcal{I}_\Xi$ is a $\mathbb{P}^{n-1}$ functor with $H = [-2]$. Here, $\Xi \subset X \times X^{[n]}$ is the universal family of length n subschemes. We denote the associated $\mathbb{P}^{n-1}$-twist by a and in case $n = 2$ the spherical twist by $\sqrt{a}$.

Lemma 5.7. *For every point $[\xi] \in X^{[n]} \setminus \partial X^{[n]}$, i.e. $\xi = \{x_1, \dots, x_n\}_{red}$ with pairwise distinct x_i, the object $a(\mathbb{C}([\xi]))$ is supported on the whole $X^{[n]}$. In case $n = 2$ the same holds for the object $\sqrt{a}(\mathbb{C}([\xi]))$.*

Proof. We set for short $A = \mathbb{C}([\xi])$. We will use the exact triangle of Fourier–Mukai transforms $F \to F' \to F''$ with kernels $\mathcal{P}' = \mathcal{O}_{X \times X^{[n]}}$ and $\mathcal{P}'' = \mathcal{O}_\Xi$. The right adjoints form the exact triangle $R'' \to R' \to R$ with kernels $\mathcal{Q}'' = \mathcal{O}_\Xi^\vee[2]$ and $\mathcal{Q}' = \mathcal{O}_{X \times X^{[n]}}[2]$. Over the open subset $X^{[n]} \setminus \partial X^{[n]}$, the universal family Ξ is smooth and thus on $\Xi_{|X^{[n]} \setminus \partial X^{[n]}}$ the object $\mathcal{O}_\Xi^\vee$ is a line bundle concentrated in degree 2. This yields

$$R''(A) = \mathcal{O}_\xi[0] \quad , \quad R'(A) = \mathrm{H}^*(X^{[n]}, A) \otimes \mathcal{O}_X[2] = \mathcal{O}_X[2]\,.$$

Setting $H^i = \mathcal{H}^i(R(A))$ the long exact cohomology sequence gives $H^{-2} = \mathcal{O}_X$, $H^{-1} = \mathcal{O}_\xi$, and $H^i = 0$ for all other values of i. The only functor in the composition $F = \mathrm{pr}_{X^{[n]}*}(\mathrm{pr}_X^*(\_) \otimes \mathcal{I}_\Xi)$ that needs to be derived is the push-forward along $\mathrm{pr}_{X^{[n]}}$. The reason is that the non-derived functors pr_X^* as well as $\mathrm{pr}_X^*(\_) \otimes \mathcal{O}_\Xi$ are exact (see [Sca09b, proposition 2.3] for the latter). Thus, there is the spectral sequence

$$E_2^{p,q} = \mathcal{H}^p(F(H^q)) \Longrightarrow E^n = \mathcal{H}^n(FR(A))$$

associated to the derived functor $\mathrm{pr}_{X^{[n]}*}$. It is zero outside of the -1 and -2 row. Now $F'(\mathcal{O}_\xi) = \mathrm{H}^*(X, \mathcal{O}_\xi) \otimes \mathcal{O}_{X^{[n]}} = \mathcal{O}_{X^{[n]}}^{\oplus n}[0]$ and $F''(\mathcal{O}_\xi)$ is also concentrated in degree zero since

Ξ is finite over $X^{[n]}$. By the long exact sequence we see that all terms in the -1 row except for $E_2^{0,-1}$ and $E_2^{1,-1}$ must vanish. Furthermore,

$$F'(H^{-2}) = \mathrm{H}^*(X, \mathcal{O}_X) \otimes \mathcal{O}_{X^{[n]}} = \mathcal{O}_{X^{[n]}}[0] \oplus \mathcal{O}_{X^{[n]}}[-2]$$

and $F''(H^{-2})$ is a locally free sheaf of rank n concentrated in degree zero since Ξ is flat of degree n over $X^{[n]}$. This shows that the -2 row of E_2 is zero outside of degree 0, 1, and 2 and that $E_2^{1,-2}$ is of positive rank. By the positioning of the non-zero terms it follows that $E_2^{1,-2} = E_\infty^{1,-2}$ and thus also $E^{-1} = \mathcal{H}^{-1}(FR(A))$ is of positive rank. Furthermore, we can read off the spectral sequence that the cohomology of $FR(A)$ is concentrated in the degrees -2, -1, and 0. Now, by the long exact sequences associated to the cones occurring in the definition of the spherical respectively the $\mathbb{P}^n$-twist it follows that $\mathcal{H}^{-2}(\sqrt{a}(A))$ as well as $\mathcal{H}^{-2}(a(A))$ are of positive rank. $\square$

Proposition 5.8. (i) *The subgroup H generated by a and push-forwards along natural automorphisms, i.e. autoequivalences of the form $\varphi_*^{[n]} = \alpha(\varphi_*)$, is isomorphic to $\mathbb{Z} \times \mathrm{Aut}(X)$.*

(ii) *$b \notin H = \langle a, \{\varphi_*^{[n]}\}_{\varphi \in \mathrm{Aut}(X)} \rangle$.*

(iii) *$a \notin G = \langle b, [\ell], \alpha(\mathrm{DAut}_{st}(X)) \rangle$.*

The same results hold for a replaced by $\sqrt{a}$ and b replaced by $\sqrt{b}$.

Proof. We have for $\varphi \in \mathrm{Aut}(X)$ that $\varphi_*^{[n]} \circ F_a = F_a \circ \varphi_*$ which by lemma 2.3 shows that a commutes with $\varphi_*^{[n]}$. The reason is that the subvariety $\Xi \subset X \times X^{[n]}$ is invariant under the morphism $\varphi \times \varphi^{[n]}$. Because of $a^k \circ \varphi_*^{[n]}(F_a(A)) = F_a(\varphi_*^k A)[k2(n-1)]$ for $A \in \mathrm{D}^b(X)$, there are no further relations in the group H which shows (i). The autoequivalence $g = a^k \circ \varphi_*^{[n]} \in H$ has $g(F(\mathcal{O}_X)) = F(\mathcal{O}_X)[2k(n-1)]$. Thus, by remark 2.6 the equality $b = g$ implies $k = 1$. But also $b = a \circ \varphi_*^{[n]}$ can not hold comparing the values of both sides on $\mathbb{C}([\xi])$ for $[\xi] \in X^{[n]} \setminus \partial X^{[n]}$. The assertion (iii) also is shown by comparing the values of the autoequivalences on $\mathbb{C}([\xi])$. $\square$

Using the same arguments as in [Add11, p.11-12 and p.39-40] one can also show that b does not equal a shift of an autoequivalence induced by a $\mathbb{P}^n$-object on $X^{[n]}$ or of an autoequivalence of the form $\alpha(T_E)$ for a spherical twist T_E on the surface. In particular, b is an exotic autoequivalence in the sense of [PS12].

6. $\mathbb{P}^n$-objects on generalised Kummer varieties

Let A be an abelian surface. There is the summation map

$$\Sigma \colon A^n \to A \quad , \quad (a_1, \dots, a_n) \mapsto \sum_{i=1}^n a_i \,.$$

We set $N_{n-1}A := \Sigma^{-1}(0)$. It is isomorphic to A^{n-1} via e.g. the morphism

$$A^{n-1} \to N_{n-1}A \quad , \quad (a_1, \dots, a_{n-1}) \mapsto (a_1, \dots, a_{n-1}, -\sum_{i=1}^{n-1} a_i) \,.$$

The subvariety $N_{n-1}A \subset A^n$ is $\mathfrak{S}_n$-invariant. Thus, we have $N_{n-1}A/\mathfrak{S}_n \subset S^n A$. The *generalized Kummer variety* is defined as $K_{n-1}A := \mu^{-1}(N_{n-1}A/\mathfrak{S}_n)$, i.e it is the subvariety of the Hilbert scheme $A^{[n]}$ consisting of all points representing subschemes whose weighted support

adds up to zero. It can be identified with $\mathrm{Hilb}^{\mathfrak{S}_n}(N_{n-1}A)$ and also the other assumptions of the Bridgeland–King–Reid theorem are satisfied which leads to the equivalence

$$\Phi = \mathrm{FM}_{\mathcal{O}_{\overline{I^nA}}} \colon \mathrm{D}^b(K_{n-1}A) \to \mathrm{D}^b_{\mathfrak{S}_n}(N_{n-1}A)$$

where $\overline{I^nX} = p^{-1}(N_{n-1}A)$ (see e.g. [Nam02]). The intersection between the small diagonal $\Delta = \delta(A) \subset A^n$ and $N_{n-1}A$ consists exactly of the points $\delta(a) = (a, \dots, a)$ for a an n-torsion point of A, i.e. $\Delta \cap N_{n-1}A = \delta(A_n)$. The intersection is transversal, since under the identification $T_{A^n} \cong T_A^{\oplus n}$ the tangent space of Δ in a point $\delta(a)$ with $a \in A_n$ is given by vectors of the form $(v, \dots, v) \in T_A(a)^{\oplus n}$ whereas the tangent space of $N_{n-1}A$ is given by vectors $(v_1, \dots, v_n) \in T_A(a)^{\oplus n}$ with $\sum_{i=1}^n v_i = 0$. Thus, we have for the tangent space of $N_{n-1}A$ in $\delta(a)$ the identification $T_{N_{n-1}A}(\delta(a)) \cong N_{\Delta/A^n}(\delta(a))$. Since the $\mathfrak{S}_n$-action on $N_{n-1}A$ is just the restriction of the action on A^n, this isomorphism is equivariant.

Theorem 6.1. *Let $n \geq 2$. For every n-torsion point $a \in A_n$ the skyscraper sheaf $\mathbb{C}(\delta(a))$ is a $\mathbb{P}^n$-object in $\mathrm{D}^b_{\mathfrak{S}_n}(N_{n-1}A)$.*

Proof. Indeed, using the results for the invariants of $\wedge^* N_{\Delta/A^n}$ of section 3

$$\begin{aligned}\mathrm{Hom}^*_{\mathrm{D}^b_{\mathfrak{S}_n}}(\mathbb{C}(\delta(a)), \mathbb{C}(\delta(a))) \cong \mathrm{Ext}^*(\mathbb{C}(\delta(a)), \mathbb{C}(\delta(a)))^{\mathfrak{S}_n} &\cong \wedge^* T_{N_{n-1}A}(\delta(a))^{\mathfrak{S}_n} \\ &\cong \wedge^* N_{\Delta/A^n}(\delta(a))^{\mathfrak{S}_n} \\ &\cong \mathbb{C} \oplus \mathbb{C}[-2] \oplus \dots \oplus \mathbb{C}[-2n] \,.\end{aligned}$$

□

Remark 6.2. For two different n-torsion points the skyscraper sheaves are orthogonal which makes the associated twists commute. Thus, we have an inclusion (see corollary 2.4)

$$\mathbb{Z}^{n^4} \subset \mathrm{Aut}(\mathrm{D}^b_{\mathfrak{S}_n}(N_{n-1}A)) \cong \mathrm{Aut}(\mathrm{D}^b(K_{n-1}A)) \,.$$

In the case $n = 2$ the generalised Kummer variety $K_{n-1}A = K_1A$ is just the Kummer surface $K(A)$. Moreover, there is an isomorphism of commutative diagrams

$$\begin{array}{ccc} \overline{I^nA} & \xrightarrow{p} & N_1A \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ K_1A & \xrightarrow[\mu]{} & N_1A/\mathfrak{S}_2 \end{array} \quad \cong \quad \begin{array}{ccc} \tilde{A} & \xrightarrow{p} & A \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ K(A) & \xrightarrow[\mu]{} & A/\iota \end{array}$$

where p and μ in the right-hand diagram are the blow-ups of the 16 different 2-torsion points respectively of their image under the quotient under the involution $\iota = (-1)$. For a 2-torsion point $a \in A_2$ we denote by $E(a)$ the exceptional divisor over the point $[a] \in A/\iota$. Since $E(a)$ is a rational curve in the $K3$-surface $K(A)$, every line bundle on it is a spherical object in $\mathrm{D}^b(K(A))$.

Proposition 6.3. *For every 2-torsion point $\Phi^{-1}(\mathbb{C}(\delta(a))) = \mathcal{O}_{E(a)}(-1)$ holds.*

Proof. Using the isomorphism of the commutative diagrams above the proof is nearly the same as the proof of proposition 4.2. □

There is no known homomorphism $\mathrm{Aut}(\mathrm{D}^b(A)) \to \mathrm{Aut}(\mathrm{D}^b(K_nA))$ analogous to Ploog's map α. But at least one can lift line bundles $L \in \mathrm{Pic}\, A$ (by restricting $\mathcal{D}_L$) and group automorphisms $\varphi \in \mathrm{Aut}(A)$ (by restricting $\varphi^{[n]}$) to the generalised Kummer variety. Recently,

Meachan has shown in [Mea12] that the restriction of Addington's functor to the generalised Kummer variety $K_n(A)$ for $n \geq 2$ (i.e. the Fourier–Mukai transform with kernel the universal sheaf) is still a $\mathbb{P}^{n-1}$ functor and thus yields an autoequivalence $\bar{a}$. Comparing these autoequivalences with those induced by the above $\mathbb{P}^n$-objects one gets results similar to the results of section 5.

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